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On the Hyper Order of Solutions of Linear Differential Equations with Entire Coefficients

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Abstract. In this paper, we investigate higher order homogeneous linear differential equations with entire coefficients of finite order. We improve and extend the results due to the second author and Hamouda by introducing the concept of hyper-order. We also consider nonhomogeneous linear differential equations.

1. Introduction

In this paper, we shall use the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [13]). In addition, we use the notations $\sigma(f)$ and $\mu(f)$ to denote respectively the order and the lower order of growth of a meromorphic function $f(z)$ and $\lambda(f)$ to denote the exponent of convergence of zeros of $f(z)$.

We define the linear measure of a set $E \subset [0, +\infty)$ by $m(E) = \int_0^{+\infty} \chi_E(t)dt$ and the logarithmic measure of a set $H \subset [1, +\infty)$ by $lm(H) = \int_1^{+\infty} \frac{\chi_H(t)}{t}dt$, where χ_F denote the characteristic function of a set F.

Definition 1.1 ([6, 22])**.** Let $f(z)$ be a meromorphic function. Then the hyper-order of $f(z)$ is defined by

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}, \tag{1.1}$$

where $T(r,f)$ is the characteristic function of Nevanlinna.

Definition 1.2 ([6])**.** Let $f(z)$ be a meromorphic function. Then the hyper-exponent of convergence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r}, \tag{1.2}$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in the disc $\{z : |z| < r\}$.

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Let $n \geq 2$ be an integer and let $A_0(z), \ldots, A_{n-1}(z)$ with $A_0(z) \not\equiv 0$ be entire functions. It is well-known that if some of the coefficients of the linear differential equation

$$f^{(n)} + A_{n-1}(z) f^{(n-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{1.3}$$

are transcendental, then (1.3) has at least one solution of infinite order. Thus a natural question arises: What conditions on $A_0(z), \ldots, A_{n-1}(z)$ will guarantee that every solutions $f \not\equiv 0$ of (1.3 is of infinite order? For the above question, there are different results for the second and higher order linear differential equations (see for example $[2-4, 6, 8-10, 12, 14-17, 19]$).

In [3], the second author and Hamouda have considered equation (1.3) and proved the following result:

Theorem A (cite3). *Let $A_0(z), \ldots, A_{n-1}(z)$ with $A_0(z) \not\equiv 0$ be entire functions. Suppose that there exist a sequence of complex numbers $(z_k)_{k\in\mathbb{N}}$ with $\lim_{k\to+\infty} z_k = \infty$ and three real numbers α, β and μ satisfying $0 \leq \beta < \alpha$ and $\mu > 0$ such that*

$$|A_0(z_k)| \geq \exp\{\alpha |z_k|^{\mu}\} \tag{1.4}$$

and

$$|A_j(z_k)| \leq \exp\{\beta |z_k|^{\mu}\} \quad (j = 1, 2, ..., n-1) \tag{1.5}$$

as $k \to +\infty$. Then every solution $f \not\equiv 0$ of the equation (1.3) has an infinite order.

For an integer $n \geq 2$, we consider the linear differential equation

$$A_n(z) f^{(n)} + A_{n-1}(z) f^{(n-1)} + ... + A_1(z) f' + A_0(z) f = 0, \tag{1.6}$$

where $A_0(z), \ldots, A_{n-1}(z), A_n(z)$ with $A_0(z) \not\equiv 0$ and $A_n(z) \not\equiv 0$ are entire functions. If $A_n \equiv 1$, it is well-known that all solutions of (1.6) are entire functions but in the case when A_n is a nonconstant entire function, it follows that the equation (1.6) can have meromorphic solutions.

Now the question which arises is: how to describe precisely the properties of growth of solutions of the equation (1.6)? Recently, L. Z. Yang [21] has considered equation (1.6) and obtained different results concerning the growth of its solutions. In [20], J. Xu and Z. Zhang have studied the equation (1.6) and obtained the following result, but the condition that the poles of every meromorphic solution of (1.6) must be of uniformly bounded multiplicity was missing. Here we give the full result:

Theorem B ([20]). *Let H be a set of complex numbers satisfying $\overline{den}\{|z| : z \in H\} > 0$, and let $A_0(z), \ldots, A_{n-1}(z), A_n(z)$ with $A_0(z) \not\equiv 0$ be entire functions such that $\max\{\sigma(A_j) \ (j = 1, 2, \ldots, n)\} \leq \sigma(A_0) = \sigma < +\infty$, and for real constants α, β satisfying $0 \leq \beta < \alpha$ and for $\varepsilon > 0$ sufficiently small, we have*

$$|A_0(z)| \geq \exp\left\{\alpha |z|^{\sigma-\varepsilon}\right\} \tag{1.7}$$

and

$$|A_j(z)| \le \exp\left\{\beta |z|^{\sigma-\varepsilon}\right\} \quad (j = 1, 2, \dots, n) \tag{1.8}$$

as $z \to \infty$ for $z \in H$. Then every meromorphic solution whose poles are of uniformly bounded multiplicity (or entire solution) $f \not\equiv 0$ of the equation (1.6) has an infinite order and satisfies $\sigma_2(f) = \sigma$.

2. Preliminary Lemmas

Lemma 2.1 ([11] p. 89). *Let $f(z)$ be a transcendental meromorphic function of finite order σ. Let $\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_m, j_m)\}$ denote a set of distinct pairs of integers satisfying $k_i > j_i \ge 0$ $(i = 1, 2, \dots, m)$ and let $\varepsilon > 0$ be a given constant. Then there exists a subset $E_1 \subset (1, +\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$ and for all $(k, j) \in \Gamma$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.1}$$

Lemma 2.2 ([11]). *Let $f(z)$ be a transcendental meromorphic function. Let $\alpha > 1$ and $\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_m, j_m)\}$ denote a set of distinct pairs of integers satisfying $k_i > j_i \ge 0$ $(i = 1, 2, \dots, m)$. Then there exist a set $E_2 \subset (1, +\infty)$ having finite logarithmic measure and a constant $B > 0$ that depends only on α and Γ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$ and all $(k, j) \in \Gamma$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le B\left[\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right]^{k-j}. \tag{2.2}$$

Lemma 2.3 ([6]). *Let $g(z)$ be an entire function of infinite order with the hyper-order $\sigma_2(g) = \sigma < +\infty$ and let $\nu_g(r)$ be the central index of $g(z)$. Then*

$$\limsup_{r\to+\infty} \frac{\log\log \nu_g(r)}{\log r} = \sigma. \tag{2.3}$$

Lemma 2.4 ([11]). *Let $f(z)$ be a meromorphic function, let j be a positive integer, and let $\alpha > 1$ be a real constant. Then there exists a constant $R > 0$ such that for all $r \ge R$, we have*

$$T\left(r, f^{(j)}\right) \le (j+2)\, T(\alpha r, f). \tag{2.4}$$

Lemma 2.5 ([7]). *Let $f(z) = g(z)/d(z)$, where $g(z)$ is a transcendental entire function with $\mu(g) = \mu(f) = \mu \le \sigma(g) = \sigma(f) \le +\infty$, and $d(z)$ is the canonical product (or polynomial) formed with the non-zero poles of $f(z)$ with $\sigma(d) = \lambda(d) = \lambda\left(\frac{1}{f}\right) = \beta < \mu$. Let z be a point with $|z| = r$ at* *which* $|g(z)| = M(r,g)$ *and* $\nu_g(r)$ *denote the central index of* g. *Then the estimation*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1+o(1)), \quad (n \geq 1 \text{ is an integer }) \tag{2.5}$$

holds for all $|z| = r \notin E_3$, *where* E_3 *is a subset of finite logarithmic measure.*

Lemma 2.6 ([7]). *Let* $f(z) = g(z)/d(z)$, *where* $g(z)$ *is a transcendental entire function with* $\mu(g) = \mu(f) = \mu \leq \sigma(g) = \sigma(f) \leq +\infty$, *and* $d(z)$ *is the canonical product (or polynomial) formed with the non-zero poles of* $f(z)$ *with* $\sigma(d) = \lambda(d) = \lambda\left(\frac{1}{f}\right) = \beta < \mu$. *Then there exists a set* $E_4 \subset (1,+\infty)$ *that has finite logarithmic measure such that for all* z *satisfying* $|z| = r \notin [0,1] \cup E_4$ *and* $|g(z)| = M(r,g)$, *we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s} \ \ (s \geq 1 \text{ is an integer}). \tag{2.6}$$

Lemma 2.7 ([5]). *Let* $g(z)$ *be a meromorphic function of order* $\sigma(g) = \alpha < +\infty$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_5 \subset (1,+\infty)$ *that has finite logarithmic measure such that for all* z *satisfying* $|z| = r \notin [0,1] \cup E_5$, $r \to +\infty$, *we have*

$$|g(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{2.7}$$

Combining Lemma 2.7 and applying it to $1/g(z)$, we obtain the following lemma.

Lemma 2.8. *Let* $g(z)$ *be a meromorphic function of order* $\sigma(g) = \alpha < +\infty$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_6 \subset (1,+\infty)$ *that has finite logarithmic measure such that for all* z *satisfying* $|z| = r \notin [0,1] \cup E_6, r \to +\infty$, *we have*

$$\exp\left\{-r^{\alpha+\varepsilon}\right\} \leq |g(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{2.8}$$

To avoid some problems caused by the exceptional set, we recall the following lemmas.

Lemma 2.9 ([12]). *Let* $\varphi : [0,+\infty) \to \mathbb{R}$ *and* $\psi : [0,+\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $\varphi(r) \leq \psi(r)$ *for all* $r \notin E_7 \cup [0,1]$, *where* $E_7 \subset (1,+\infty)$ *is a set of of finite logarithmic measure. Let* $\alpha > 1$ *be a given constant. Then there exists an* $r_0 = r_0(\alpha) > 0$ *such that* $\varphi(r) \leq \psi(\alpha r)$ *for all* $r > r_0$.

Lemma 2.10 ([1]). *Let* $g : [0,+\infty) \to \mathbb{R}$ *and* $h : [0,+\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $g(r) \leq h(r)$ *outside of an exceptional set* $E_8 \subset (0,+\infty)$ *of finite linear measure. Then for any* $\lambda > 1$, *there exists* $r_1 > 0$ *such that* $g(r) \leq h(\lambda r)$ *for all* $r > r_1$.

3. Main Results

The main purpose of this paper is to improve and extend Theorem A for equations of the form (1.6) by using the concept of hyper-order and considering some coefficient A_s $(s = 0, 1, \ldots, n-1)$. We shall prove the following results.

Theorem 3.1. *Let $A_0(z), \ldots, A_{n-1}(z), A_n(z)$ be entire functions with $A_0(z) \not\equiv 0$ and $A_n(z) \not\equiv 0$ such that there exists some integer s $(s = 0, 1, \ldots, n-1)$ satisfying*

$$\max\{\sigma(A_j) \quad (j \neq s)\} < \mu(A_s) \leq \sigma(A_s) = \sigma < +\infty.$$

Suppose that there exist a sequence of complex numbers $(z_k)_{k\in\mathbb{N}}$ with $\lim_{k\to+\infty} z_k = \infty$ and two real numbers α and β satisfying $0 \leq \beta < \alpha$ such that for $\varepsilon > 0$ sufficiently small, we have

$$|A_s(z_k)| \geq \exp\left\{\alpha |z_k|^{\sigma-\varepsilon}\right\} \tag{3.1}$$

and

$$|A_j(z_k)| \leq \exp\left\{\beta |z_k|^{\sigma-\varepsilon}\right\} \quad (j \neq s) \tag{3.2}$$

as $k \to +\infty$. Then every transcendental meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicity of the equation (1.6) has an infinite order and satisfies $\sigma_2(f) = \sigma$.

Proof. Set

$$\max\{\sigma(A_j) \quad (j \neq s)\} = \lambda < \mu(A_s) \leq \sigma(A_s) = \sigma < +\infty. \tag{3.3}$$

Let f $(\not\equiv 0)$ be a transcendental meromorphic solution whose poles are of uniformly bounded multiplicity of (1.6). We set $f(z) = g(z)/d(z)$, where $g(z)$ is an entire function and $d(z)$ is the canonical product (or polynomial) formed with the non-zero poles of $f(z)$. By the fact that the poles of $f(z)$ can only occur at the zeros of $A_n(z)$, it follows that $\sigma(d) = \lambda(d) = \lambda\left(\frac{1}{f}\right) \leq \lambda < \mu(A_s)$.

First assume that $\sigma(f) = \rho < +\infty$. For $j = 0, ..., n-1$, since

$$T\left(r, f^{(j+1)}\right) \leq 2T\left(r, f^{(j)}\right) + m\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right),$$

$$m\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right) = O(\log r),$$

we can obtain by using Lemma 2.4 for all $r \geq R$

$$\begin{aligned} T\left(r, f^{(j+1)}\right) &\leq 2T\left(r, f^{(j)}\right) + O(\log r) \\ &\leq 2(j+2)\,T(2r, f) + O(\log r). \end{aligned} \tag{3.4}$$

We can rewrite (1.6) as

$$-A_s(z) = A_n(z)\frac{f^{(n)}}{f^{(s)}} + A_{n-1}(z)\frac{f^{(n-1)}}{f^{(s)}} + \cdots + A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}}$$

$$+A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}} + \cdots + A_1(z)\frac{f'}{f^{(s)}} + A_0(z)\frac{f}{f^{(s)}}. \tag{3.5}$$

By (3.4) and (3.5), we obtain for all $r \geq R$

$$T(r, A_s) \leq cT(2r, f) + \sum_{j\neq s} T(r, A_j) + O(\log r), \tag{3.6}$$

where c (> 0) is a constant. By (3.6) and (3.3), we conclude that $\mu(f) \geq \mu(A_s)$. By the fact that $\sigma(d) = \lambda(d) = \lambda\left(\frac{1}{f}\right) \leq \lambda < \mu(A_s)$ and the inequality $T(r,f) \leq T(r,g) + T(r,d) + O(1)$, it follows that $\mu(g) = \mu(f) \geq \mu(A_s) > \lambda \geq \sigma(d)$ and $\sigma(g) = \sigma(f) < +\infty$. Hence by Lemma 2.6, there exists a set $E_4 \subset (1, +\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_4$ and $|g(z)| = M(r,g)$, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s} \ \ (s \geq 1 \text{ is an integer}). \tag{3.7}$$

By Lemma 2.1, there exists a subset $E_1 \subset (1, +\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1 \cup [0,1]$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leq r^{(j-s)(\rho-1+\varepsilon)} \quad (j = s+1, \ldots, n) \tag{3.8}$$

and

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq r^{j(\rho-1+\varepsilon)} \ (j = 1, \ldots, s-1). \tag{3.9}$$

We can rewrite (1.6) as

$$\begin{aligned} &\frac{A_n(z)}{A_s(z)}\frac{f^{(n)}}{f^{(s)}} + \frac{A_{n-1}(z)}{A_s(z)}\frac{f^{(n-1)}}{f^{(s)}} + \cdots + \frac{A_{s+1}(z)}{A_s(z)}\frac{f^{(s+1)}}{f^{(s)}} \\ &+ \frac{A_{s-1}(z)}{A_s(z)}\frac{f^{(s-1)}}{f}\frac{f}{f^{(s)}} + \cdots + \frac{A_1(z)}{A_s(z)}\frac{f'}{f}\frac{f}{f^{(s)}} + \frac{A_0(z)}{A_s(z)}\frac{f}{f^{(s)}} = -1. \end{aligned} \tag{3.10}$$

From (3.1), (3.2) and (3.7)-(3.9), we have

$$\left|\frac{A_j(z_k)}{A_s(z_k)}\right|\left|\frac{f^{(j)}(z_k)}{f^{(s)}(z_k)}\right| \leq \frac{|z_k|^{(j-s)(\rho-1+\varepsilon)}}{\exp\left\{(\alpha-\beta)|z_k|^{\sigma-\varepsilon}\right\}} \ \ (j = s+1, \ldots, n), \tag{3.11}$$

(3.12)

$$\left|\frac{A_j(z_k)}{A_s(z_k)}\right|\left|\frac{f^{(j)}(z_k)}{f(z_k)}\right|\left|\frac{f(z_k)}{f^{(s)}(z_k)}\right| \leq \frac{|z_k|^{2s+j(\rho-1+\varepsilon)}}{\exp\left\{(\alpha-\beta)|z_k|^{\sigma-\varepsilon}\right\}} \ \ (j = 1, \ldots, s-1)$$

and

$$\left|\frac{A_0(z_k)}{A_s(z_k)}\right|\left|\frac{f(z_k)}{f^{(s)}(z_k)}\right| \leq \frac{|z_k|^{2s}}{\exp\left\{(\alpha-\beta)|z_k|^{\sigma-\varepsilon}\right\}}, \tag{3.13}$$

where $|z_k| = r_k \notin [0,1] \cup E_1 \cup E_4$ and $|g(z_k)| = M(r_k, g)$. From (3.11)-(3.13), it follows that

$$\lim_{k\to+\infty}\left|\frac{A_j(z_k)}{A_s(z_k)}\right|\left|\frac{f^{(j)}(z_k)}{f^{(s)}(z_k)}\right| = 0 \ \ (j = s+1,\dots,n),$$

$$\lim_{k\to+\infty}\left|\frac{A_j(z_k)}{A_s(z_k)}\right|\left|\frac{f^{(j)}(z_k)}{f(z_k)}\right|\left|\frac{f(z_k)}{f^{(s)}(z_k)}\right| = 0 \ \ (j = 1,\dots,s-1)$$

and

$$\lim_{k\to+\infty}\left|\frac{A_0(z_k)}{A_s(z_k)}\right|\left|\frac{f(z_k)}{f^{(s)}(z_k)}\right| = 0.$$

By making $k \to +\infty$ in relation (3.10), we get a contradiction. Hence $\sigma(f) = +\infty$.

From (3.5), it follows that

$$|A_s(z)| \leq |A_n(z)|\left|\frac{f^{(n)}}{f^{(s)}}\right| + |A_{n-1}(z)|\left|\frac{f^{(n-1)}}{f^{(s)}}\right| + \cdots + |A_{s+1}(z)|\left|\frac{f^{(s+1)}}{f^{(s)}}\right|$$

$$+|A_{s-1}(z)|\left|\frac{f^{(s-1)}}{f}\right|\left|\frac{f}{f^{(s)}}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right|\left|\frac{f}{f^{(s)}}\right| + |A_0(z)|\left|\frac{f}{f^{(s)}}\right|. \tag{3.14}$$

By Lemma 2.2, there exist a constant $B > 0$ and a set $E_2 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2 \cup [0,1]$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leq Br\,[T(2r,f)]^{j-s+1} \quad (j = s+1,\dots,n), \tag{3.15}$$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq Br\,[T(2r,f)]^{j+1} \quad (j = 1,\dots,s-1). \tag{3.16}$$

Hence from (3.1), (3.2), (3.7) and (3.14)-(3.16), it follows that

$$\exp\left\{\alpha|z_k|^{\sigma-\varepsilon}\right\} \leq Bn|z_k|^{2s+1}[T(2r_k,f)]^{n+1}\exp\left\{\beta|z_k|^{\sigma-\varepsilon}\right\} \tag{3.17}$$

as $r_k \to +\infty$, $|z_k| = r_k \notin [0,1] \cup E_2 \cup E_4$ and $|g(z_k)| = M(r_k, g)$. By Lemma 2.9 and (3.17), it follows that $\sigma_2(f) \geq \sigma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\sigma_2(f) \geq \sigma$.

Now we prove that $\sigma_2(f) \leq \sigma$. We can rewrite (1.6) as

$$-A_n(z)\frac{f^{(n)}}{f} = A_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + A_{s+1}(z)\frac{f^{(s+1)}}{f}$$

$$+A_s(z)\frac{f^{(s)}}{f}+A_{s-1}(z)\frac{f^{(s-1)}}{f}+\cdots+A_1(z)\frac{f'}{f}+A_0(z). \tag{3.18}$$

By Lemma 2.5, there exist a set $E_3\subset(1,+\infty)$ of finite logarithmic measure such that for all z satisfying $|z|=r\notin E_3$ and $|g(z)|=M(r,g)$, we have

$$\frac{f^{(n)}(z)}{f(z)}=\left(\frac{\nu_g(r)}{z}\right)^n(1+o(1))\quad(n\geq 1\quad\text{is an integer}). \tag{3.19}$$

By Lemma 2.8, there exists a set $E_6\subset(1,+\infty)$ that has finite linear measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_6$, $r\to+\infty$, we have

$$|A_j(z)|\leq\exp\left\{r^{\sigma+\varepsilon}\right\}\quad(j=0,1,\ldots,n-1) \tag{3.20}$$

and

$$|A_n(z)|\geq\exp\left\{-r^{\sigma+\varepsilon}\right\}. \tag{3.21}$$

Substituting (3.19) into (3.18), for all z satisfying $|z|=r\notin E_3$ and $|g(z)|=M(r,g)$, we have

$$\begin{aligned}-A_n(z)\left(\frac{\nu_g(r)}{z}\right)^n(1+o(1))&=A_{n-1}(z)\left(\frac{\nu_g(r)}{z}\right)^{n-1}(1+o(1))\\
+\cdots+A_{s+1}(z)\left(\frac{\nu_g(r)}{z}\right)^{s+1}&(1+o(1))+A_s(z)\left(\frac{\nu_g(r)}{z}\right)^{s}(1+o(1))\\
+A_{s-1}(z)\left(\frac{\nu_g(r)}{z}\right)^{s-1}(1+o(1))&+\cdots+A_1(z)\left(\frac{\nu_g(r)}{z}\right)(1+o(1))+A_0(z).\end{aligned} \tag{3.22}$$

Hence from (3.20)-(3.22), for all z satisfying $|z|=r\notin[0,1]\cup E_3\cup E_6$, $r\to+\infty$ and $|g(z)|=M(r,g)$, we have

$$\begin{aligned}\exp\left\{-r^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r)}{z}\right|^n|1+o(1)|&\leq\exp\left\{r^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r)}{z}\right|^{n-1}|1+o(1)|\\
&+\cdots+\exp\left\{r^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r)}{z}\right||1+o(1)|+\exp\left\{r^{\sigma+\varepsilon}\right\}\\
&\leq n\exp\left\{r^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r)}{z}\right|^{n-1}|1+o(1)|.\end{aligned} \tag{3.23}$$

By (3.23) and Lemma 2.9, we get

$$\limsup_{r\to+\infty}\frac{\log\log\nu_g(r)}{\log r}\leq\sigma+\varepsilon. \tag{3.24}$$

Since $\varepsilon>0$ is arbitrary, by (3.24) and Lemma 2.3, we obtain $\sigma_2(g)\leq\sigma$. Hence $\sigma_2(f)\leq\sigma$. This and the fact that $\sigma_2(f)\geq\sigma$ yield $\sigma_2(f)=\sigma$. □

Considering the nonhomogeneous linear differential equation, we obtain:

Theorem 3.2. *Let $A_0(z), \dots, A_{n-1}(z)$, $A_n(z)$ with $A_0(z) \not\equiv 0$ and $A_n(z) \not\equiv 0$ be entire functions satisfying the hypotheses of Theorem 3.1 and let $F \not\equiv 0$ be an entire function such that $\sigma(F) < \mu(A_s)$ Then every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicity of the linear differential equation*

$$A_n(z) f^{(n)} + A_{n-1}(z) f^{(n-1)} + \cdots + A_1(z) f' + A_0(z) f = F \tag{3.25}$$

satisfies $\overline{\lambda}_2(f) = \sigma_2(f) = \sigma$, with at most one exceptional solution f_0 of finite order.

Proof. First we show that (3.25) can possess at most one exceptional solution f_0 of finite order. In fact, if f^* is another solution of finite order of equation (3.25), then $f_0 - f^*$ is of finite order. But $f_0 - f^*$ is a solution of the corresponding homogeneous equation (1.6) of (3.25). This contradicts Theorem 3.1. We assume that f is an infinite order transcendental meromorphic solution of (3.25) and $f_1, f_2, \dots, f_n$ is a solution base of the corresponding homogeneous equation (1.6) of (3.25). Then f can be expressed in the form

$$f(z) = B_1(z) f_1(z) + B_2(z) f_2(z) + \cdots + B_n(z) f_n(z), \tag{3.26}$$

where $B_1(z), \dots, B_n(z)$ are suitable meromorphic functions determined by (3.27)

$$\begin{array}{c} B_1'(z) f_1(z) + B_2'(z) f_2(z) + ... + B_n'(z) f_n(z) = 0 \\ B_1'(z) f_1'(z) + B_2'(z) f_2'(z) + ... + B_n'(z) f_n'(z) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ B_1'(z) f_1^{(n-1)}(z) + B_2'(z) f_2^{(n-1)}(z) + ... + B_n'(z) f_n^{(n-1)}(z) = F(z). \end{array}$$

Since the Wronskian $W(f_1, f_2, \dots, f_n)$ is a differential polynomial in $f_1, f_2, \dots, f_n$ with constant coefficients, it is easy to deduce that

$$\sigma_2(W) \leq \max\{\sigma_2(f_j) : j = 1, \dots, n\} = \sigma(A_s) = \sigma.$$

From (3.27), we have

$$B_j' = F G_j(f_1, f_2, \dots f_n) W(f_1, f_2, \dots, f_n)^{-1} \quad (j = 1, 2, \dots, n), \tag{3.28}$$

where $G_j(f_1, f_2, \dots f_n)$ are differential polynomials in $f_1, f_2, \dots, f_n$ with constant coefficients. Thus (3.29)

$$\sigma_2(G_j) \leq \max\{\sigma_2(f_j) : j = 1, \dots, n\} = \sigma(A_s) = \sigma \quad (j = 1, 2, \dots, n).$$

By (3.28) and (3.29), we have (3.30)

$$\sigma_2(B_j) = \sigma_2\left(B_j'\right) \leq \max\{\sigma_2(F), \sigma(A_s)\} = \sigma(A_s) \quad (j = 1, 2, \dots, n).$$

Then from (3.26) and (3.30), we get

$$\sigma_2(f) \leq \max\{\sigma_2(f_j), \sigma_2(B_j) : j = 1, 2, \dots, n\} = \sigma(A_s). \tag{3.31}$$

We can rewrite (3.25) as

$$-A_s(z) = A_n(z)\frac{f^{(n)}}{f^{(s)}} + A_{n-1}(z)\frac{f^{(n-1)}}{f^{(s)}} + \cdots + A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}}$$

$$+A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}} + \cdots + A_1(z)\frac{f'}{f^{(s)}} + A_0(z)\frac{f}{f^{(s)}} - \frac{F(z)}{f^{(s)}}. \tag{3.32}$$

Set

$$\max\{\sigma(A_j)\ \ (j \neq s), \sigma(F)\} = \gamma < \mu(A_s) \leq \sigma(A_s) = \sigma < +\infty. \tag{3.33}$$

We set $f(z) = g(z)/d(z)$, where $g(z)$ is an entire function and $d(z)$ is the canonical product (or polynomial) formed with the non-zero poles of $f(z)$. By the fact that the poles of $f(z)$ can only occur at the zeros of $A_n(z)$, it follows that $\sigma(d) = \lambda(d) = \lambda\left(\frac{1}{f}\right) \leq \gamma < \mu(A_s)$. For $j = 0, \ldots, n-1$, since

$$T\left(r, f^{(j+1)}\right) \leq 2T\left(r, f^{(j)}\right) + m\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right),$$

$$m\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right) = O\left(\log rT\left(r, f^{(j)}\right)\right),$$

we can obtain by using Lemma 2.4 for all $r \geq R$

$$T\left(r, f^{(j+1)}\right) \leq 2T\left(r, f^{(j)}\right) + O\left(\log rT\left(r, f^{(j)}\right)\right)$$

$$\leq 2(j+2)T(2r, f) + O\left(\log rT\left(r, f^{(j)}\right)\right). \tag{3.34}$$

We have also for sufficiently large r

$$O\left(\log rT\left(r, f^{(j)}\right)\right) = o\left(T\left(r, f^{(j)}\right)\right)$$

which yields

$$O\left(\log rT\left(r, f^{(j)}\right)\right) \leq \frac{1}{2}T\left(r, f^{(j)}\right). \tag{3.35}$$

By (3.34), (3.35) and Lemma 2.4, we can obtain from (3.32) for sufficiently large r

$$T(r, A_s) \leq T(r, F) + cT(2r, f) + \sum_{j \neq s} T(r, A_j), \tag{3.36}$$

where $c\,(>0)$ is a constant. By (3.36) and (3.33), we conclude $\mu(f) \geq \mu(A_s)$. By the fact that $\sigma(d) = \lambda(d) = \lambda\left(\frac{1}{f}\right) \leq \gamma < \mu(A_s)$ and the inequality $T(r, f) \leq T(r, g) + T(r, d) + O(1)$, it follows that $\mu(g) = \mu(f) > \sigma(d)$ and $\sigma(g) = \sigma(f) = +\infty$. Hence by Lemma 2.6, there exists a set $E_4 \subset (1, +\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_4$ and $|g(z)| = M(r, g)$, we have (3.7) holds. By Lemma

2.2, there exist a constant $B > 0$ and a set $E_2 \subset (1, +\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2 \cup [0,1]$, we have (3.15) and (3.16) hold. From (3.32), it follows that

$$|A_s(z)| \leq |A_n(z)| \left| \frac{f^{(n)}}{f^{(s)}} \right| + |A_{n-1}(z)| \left| \frac{f^{(n-1)}}{f^{(s)}} \right| + \cdots + |A_{s+1}(z)| \left| \frac{f^{(s+1)}}{f^{(s)}} \right|$$

$$+ |A_{s-1}(z)| \left| \frac{f^{(s-1)}}{f} \right| \left| \frac{f}{f^{(s)}} \right| + \cdots + |A_1(z)| \left| \frac{f'}{f} \right| \left| \frac{f}{f^{(s)}} \right|$$

$$+ |A_0(z)| \left| \frac{f}{f^{(s)}} \right| + \left| \frac{F}{f} \right| \left| \frac{f}{f^{(s)}} \right|. \tag{3.37}$$

On the other hand, for any given ε $(0 < 2\varepsilon < \sigma - \gamma)$, we have for a sufficiently large r

$$|F(z)| \leq \exp\left\{r^{\gamma+\varepsilon}\right\} \text{ and } |d(z)| \leq \exp\left\{r^{\gamma+\varepsilon}\right\}. \tag{3.38}$$

Since $M(r,g) \geq 1$, it follows from (3.7) and (3.38) that

$$\left| \frac{F(z)}{f(z)} \right| \left| \frac{f(z)}{f^{(s)}(z)} \right| = \frac{|F(z)|\,|d(z)|}{|g(z)|} \left| \frac{f(z)}{f^{(s)}(z)} \right| \leq r^{2s} \exp\left\{2r^{\gamma+\varepsilon}\right\} \tag{3.39}$$

as $|z| = r \to +\infty$ and $|g(z)| = M(r,g)$. From (3.1), (3.2), (3.7), (3.15), (3.16) and (3.39), it follows that

$$\exp\left\{\alpha |z_k|^{\sigma-\varepsilon}\right\} \leq Bn |z_k|^{2s+1} [T(2r_k, f)]^{n+1} \exp\left\{\beta |z_k|^{\sigma-\varepsilon}\right\}$$

$$+ |z_k|^{2s} \exp\left\{2 |z_k|^{\gamma+\varepsilon}\right\} \tag{3.40}$$

as $k \to +\infty$, $|z_k| = r_k \notin [0,1] \cup E_2 \cup E_4$ and $|g(z_k)| = M(r_k, g)$. From (3.40) and Lemma 2.9, we get $\sigma_2(f) \geq \sigma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\sigma_2(f) \geq \sigma$. This and the fact that $\sigma_2(f) \leq \sigma$ yield $\sigma_2(f) = \sigma$.

By (3.25), it is easy to see that if f has a zero z_0 of order α $(> n)$, then F must have a zero at z_0 of order $\alpha - n$. Hence

$$n\left(r, \frac{1}{f}\right) \leq n\overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right)$$

and

$$N\left(r, \frac{1}{f}\right) \leq n\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right). \tag{3.41}$$

We can rewrite (3.25) as

$$\frac{1}{f} = \frac{1}{F}\left(A_n(z)\frac{f^{(n)}}{f} + A_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z)\right). \tag{3.42}$$

By (3.42), we have

$$m\left(r,\frac{1}{f}\right)\leq\sum_{j=1}^{n}m\left(r,\frac{f^{(j)}}{f}\right)+\sum_{j=0}^{n}m\left(r,A_j\right)+m\left(r,\frac{1}{F}\right)+O\left(1\right). \tag{3.43}$$

By (3.41) and (3.43), we obtain for $|z|=r$ outside a set E of finite linear measure

$$T\left(r,f\right)=T\left(r,\frac{1}{f}\right)+O\left(1\right)$$

$$\leq n\overline{N}\left(r,\frac{1}{f}\right)+\sum_{j=0}^{n}T\left(r,A_j\right)+T\left(r,F\right)+O\left(\log\left(rT\left(r,f\right)\right)\right). \tag{3.44}$$

For sufficiently large r and any given $\varepsilon>0$, we have

$$O\left(\log r+\log T\left(r,f\right)\right)\leq\frac{1}{2}T\left(r,f\right), \tag{3.45}$$

$$\sum_{j=0}^{n}T\left(r,A_j\right)\leq\left(n+1\right)r^{\sigma+\varepsilon} \tag{3.46}$$

and

$$T\left(r,F\right)\leq r^{\sigma(F)+\varepsilon}. \tag{3.47}$$

Thus by (3.44) – (3.47), we have

$$T\left(r,f\right)\leq 2n\overline{N}\left(r,\frac{1}{f}\right)+2\left(n+1\right)r^{\sigma+\varepsilon}+2r^{\sigma(F)+\varepsilon}, \tag{3.48}$$

where $|z|=r\notin E$. Hence for any f with $\sigma_2\left(f\right)=\sigma$, by (3.48) and Lemma 2.10, we have $\sigma_2\left(f\right)\leq\overline{\lambda}_2\left(f\right)$. Therefore, $\overline{\lambda}_2\left(f\right)=\sigma_2\left(f\right)=\sigma$. □

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